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16. Abstract

Discussion of the application of variational techniques to the solution of boundary-value problems for plastic media, with particular reference to nonlinear-elastic, nonlinear-plastic, elastoplastic, and rigid-plastic bodies. The principles of the original and modified Ritz methods are outlined, showing that the modified method makes it possible to overcome the difficulties associated with the non-quadraticity of functionals and to obtain solutions in a straightforward way. The optimum-interpolation method and the variational-difference method are also explained.

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VARIATIONAL METHODS IN THE THEORY OF PLASTICITY

L. M. Kachanov

Extreme principles in the theory of plasticity are of interest for two reasons. First, important general properties of plastic equilibrium, questions of uniqueness and of the existence of a solution are explained by means of them. Second, extreme principles make possible a straightforward solution of specific problems, avoiding differential equations; this possibility is important, in view of the nonlinearity of the equations of the theory of plasticity. /177*

This explains the great interest in extreme theorems and the considerable number of publications in this field. Excellent surveys of general theorems for elastic-plastic media have been published recently by D. Drukker [1], V. Koyter [2] and R. Hill [3]. In these works however, problems of constructing solutions based on extreme principles were not touched on, although much effort has been made in this direction in recent years. Therefore, it is advisable here to dwell on the use of variational principles for formulating solutions.

It appeared at one time that development of electronic computers would permit confining oneself to the structurally simplest calculating procedures, in particular, would permit solution of boundary-value problems by the network method. Howevever, experience has not confirmed this for equations in partial derivatives, and it has showed that other methods, variational ones, are more effective.

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* Numbers in the margin indicate pagination in the foreign text.

We confine ourselves to examination of boundary-value problems for the simplest plastic media. Variational methods of solution of the problem of transient creep was examined in works [4-5], problems with finite displacements (excluding steady flow) in works [6, 3] and problems of stability beyond the elastic limit in works [7-9].

Nonlinear-Elastic and Nonlinear-Plastic Bodies

1. Nonlinear-Elastic Body

The equations of state of a nonlinear-elastic body are simple. As is well known, in simple and monotonic loading, plastic deformation can be described by means of these equations. In this case, we arrive at the equations of the theory of elastic-plastic deformations (Genki-Navai theory).

Let σ, ε be the stress and deformation tensors, with components σ_{ij}, ε_{ij}. We assume σ_{ij} are single-value functions of ε_{ij} with reciprocity relationship $\frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}$ satisfied. Then, there exist components of deformation, the deformation potential

$$\Pi = \int \sigma_{ij} d\varepsilon_{ij} = \Pi(\varepsilon). \tag{1.1}$$

in which

$$\sigma_{ij} = \frac{\partial \Pi}{\partial \varepsilon_{ij}}. \tag{1.2}$$

We will consider $\Pi(\varepsilon)$ to be a convex function. Let a force F_n be assigned to a portion S_F of the surface of body S, and on the remaining portion S_u, a displacement u. For simplicity, we do not take volumetric forces into account. We agreed to distinguish continuous displacement u' by a prime, assuming a fixed value on S_u: field u' corresponds to deformation ε'_{ij}.

Then, the minimum principle for displacement holds true: the total energy of the body $\Pi^*(\varepsilon)$ reaches an absolute minimum for the actual displacement field [3]

$$\Pi^*(\varepsilon) = \int \Phi(\varepsilon) dV - \int \ldots \qquad (1.3)$$

with respect to the entire field u'.

The extreme is analytical here [10]; therefore,

$$\delta \ldots = 0 \qquad (1.4)$$

If the components of deformation ε_{ij} are single-value functions of σ_{ij} and the corresponding reciprocity ratio holds true, there exists a stress function, supplementary work

$$\ldots \qquad (1.5)$$

in which

$$\ldots = \frac{\partial \ldots}{\partial \sigma_{ij}}. \qquad (1.6)$$

If $R(\sigma)$ is a convex function, and σ'_{ij} is an arbitrary stress field, satisfying the differential equations of equilibrium and limited conditionally to S_p, the minimum principle for stress holds true: the total supplementary work of the body $R^*(\sigma)$ reaches an absolute minimum for the actual stress field [10]:

$$R^*(\sigma) \equiv \int R(\sigma) dV - \int \ldots \qquad (1.7)$$

The extreme in (1.7) also is analytical, i.e.,

$$\Pi^* = 0 \tag{1.8}$$

If there simultaneously exist both functions Π and R, both principles are fulfilled simultaneously. Then, $\Pi + R = \sigma_{ij}\varepsilon_{ij}$ and it is easy to see that, for an actual solution, $\Pi^*(\varepsilon) = -R^*(\sigma)$.

Consequently, the following estimate holds true:

$$\Pi^* \leq R^* \tag{1.9}$$

Between the average pressure $\sigma = 1/3\ \sigma_u$ and the relative change in volume $\varepsilon = \varepsilon_{ii}$, a linear relationship usually is assumed

$$\sigma = k\varepsilon \tag{1.10}$$

where k is the modulus of bulk compression.

If the expectancy is disregarded (i.e., at k = 0), the stress deviator components s_{ij} should be substituted in the preceding relationships of the stress components.

The intensity of the tangential stress $T = (0.5 s_{ij} s_{ij})^{1/2}$, where s_{ij} are components of the stress deviator and the intensity of displacement deformation $\Gamma = (2 e_{ij} e_{ij})^{1/2}$, where e_{ij} are components of the deformation deviator, are connected by the relationship

$$T = \Phi(\Gamma)\Gamma \quad \text{or} \quad \Gamma = \bar{\varphi}(T)T, \tag{1.11}$$

the appearance of which is shown in Fig. 1.

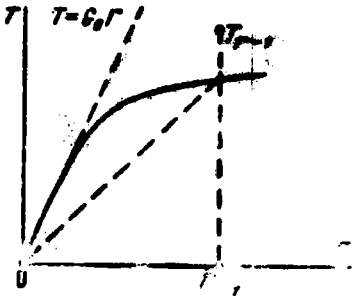

Fig. 1.

We note here the partial case of a power function, which is important for application,

$$T = B\Gamma^{\mu} \quad \text{or} \quad \Gamma = BT^{m}, \tag{1.12}$$

where B, $m \geq 1$ are constants ($\mu = 1/m$).

2. Nonlinear-Elastic Body

If the medium is considered to be incompressible, velocity v is substituted for displacement u and deformation velocity tensor ξ for deformation tensor ε, in place of (1.2) and (1.6), there will be the relationships:

$$s_{ij} = \frac{\partial L}{\partial \xi_{ij}}; \quad \xi_{ij} = \frac{\partial \Lambda}{\partial s_{ij}} \tag{2.1}$$

where L is the dispersion and Λ is the supplementary dispersion [11]. Equations of the type (1.2) are widely used in the steady creep problem.

The extreme principles introduced above are completely transposed in this case, if Π^* is replaced by L^* and R^* by Λ^*; in this case, velocity v is set on part of the surface S_v.

In this manner, there is an analogy between the solutions of the corresponding boundary-value problems for nonlinear-elastic and nonlinear-plastic bodies [12]. Considering this, we will examine below only the case of a nonlinear-elastic body.

3. Existence and Singleness of the Solution

As has already been noted, this work is not oriented towards discussion of these problems. We confine ourselves only to certain observations.

Proof of the singleness of the solution causes no difficulties and flows directly from the principles of the absolute minimum of Π^* and R^*.

Proof of the existence of a solution of boundary-value problems is more difficult. Certain heuristic considerations in connection with this were presented in the work of Koyter [2]. A strict proof of the existence of generalized solutions was given in the articles of A. Langenbach [13-14]. The limitations which must be superimposed on relationship (1.11) are of greater interest here to us. These limitations are:

$$\begin{aligned} &\bar{g}(T) \geqslant \bar{g}_0 > 0, \qquad \bar{g}_0 = \text{const}; \\ &\frac{dT}{d\Gamma} \geqslant \varkappa_1 > 0, \qquad \varkappa_1 = \text{const} \end{aligned} \tag{3.1}$$

and, correspondingly,

$$\begin{aligned} &g(\Gamma) \geqslant g_0 > 0, \qquad g_0 = \text{const} \\ &\frac{d\Gamma}{dT} \geqslant \varkappa_2 > 0, \qquad \varkappa_2 = \text{const} \end{aligned} \tag{3.2}$$

A. Langenbach proved the existence of a classical solution of the problem, for the elastic-plastic torsion problem [15].

We note that conditions (3.1) and (3.2) are not satisfied, in the vicinity of the zero stress state in the power functions (1.11) and (1.12) and $m \neq 1$. However, it can be considered that, /180

with developed deformation of a body, a solution of the problem by the power function is a good approximation to a solution of the problem, with a close dependence characterized by a "good" initial section [13].

4. Ritz Method

Let us examine variational equation (1.4) for definiteness. Let u_k be a series of coordinate functions. We seek an approximate solution of the problem of the minimum (1.4), in the form

$$u_{(n)} = \sum_{k=1}^{n} c_k u_k \qquad (4.1)$$

The coefficients c_k are determined from the condition of the minimum Π^*. In the works of A. Langenbach [13, 14] for the plastic problems being examined, the solvability of the Ritz system was established for any finite n (under certain additional conditions). Solvability of the Ritz system for functionals of a more general type was proved by L. N. Hagen-Torn and S. G. Mikhlin [16]. However, formulation and solution of the Ritz system in nonlinear problems, even at small n, involves tremendous calculation difficulties.

In connection with this, a single-value approximation (under zero conditions on S_u) of the type $u_{(1)} = c_1 u_1$ has become widespread. The solution of the corresponding linear (elastic) problem usually is used for u_1. In this form, this method is used for approximate solution of practical engineering problems (for example, the problem of plastic deformation and creep of deflectable plates, axially symmetric shells and deformed slabs). Simple longitudinal bending of a cantilever [4] by the power law is evidence of the unreliability of this method; the discrepancy between the approximate and precise solutions can be considerable here.

In other problems, especially in the problem of bending of plates, the divergence apparently is considerably less.

The method of solution with one arbitrary constant is more acceptable in cases, when the value sought is proportional to the minimizing one. For example, this concerns solution of the problem of bending a thin-walled pipe in a curve [4] with power function (1.12). In this problem, by generalization to the [word illegible] problem of Karman, the coefficient of flexibility of the curve in the pipe is proportional to the supplementary work, a value, which is directly [word illegible].

In a number of problems, simple and fortunate solutions have been successfully obtained, using one or two arbitrary constants. For example, the solution of Shi Po-ming [17] of the problem of sags of elastic-plastic [word illegible] is such a one. However, there never is confidence in the reliability of a solution with 1-2 arbitrary constants. Such solutions require comparison with other results or experimental verification.

The difficulties increase sharply with increase in number of arbitrary constants. Even in the case of a quadratic functional, solution of the Ritz system involves a large volume of analytical computations, which the Ritz method guarantees in practice. The situation is being saved by the recent appearance of methods of carrying out the necessary computations with the aid of electronic computers [18].

Still another difficulty arises with the transition to non-linear problems -- the necessity for calculation of the integrals from $\Pi^*(\varepsilon')$ (or $R^*(\sigma')$, where [word illegible] σ') contains arbitrary constants.

The deformation potential Π or supplementary work R can be transcendental or irrational functions of the intensities Γ or T,

and the difficult problem inescapably arises of approximation of these functions by sufficiently simple polynomials.

If a nonlinear Ritz system is obtained by one means or another, its solution involves considerable difficulties. The method proposed by D. F. Davidenko [19] of solution of nonlinear systems has well-known advantages; they are based on reduction of the latter to systems of ordinary differential equations and solution for them of the Cauchey problem.

Another method used by V. N. Ionov [20] consists of the following. A nonlinear Ritz system can be formally recorded as linear (for the corresponding elastic problem), with coefficients, containing factors of the type $1 + \phi(T)$, where the plasticity function $\phi(T)$ is determined from the deformation curve. In the zero approximation, we assume $\phi(T) = 0$; we find T_0 and $\phi(T_0)$ from the zero approximation. The linear system is solved again with coefficients, containing factors of the type $1 + \phi(T_0)$, etc.

5. Modified Ritz Method

The difficulties in direct application of the Ritz method to nonlinear problems impels a search for various modifications of it. One such modification was proposed by the author of [21], and it can be used in searching for the minimum in various nonlinear problems. This method permits the difficulties connected with the nonquadratic nature of the functionals to be overcome and solutions by direct methods to be formulated with the necessary accuracy.

Let us examine, for example, application of this method to the search for a minimum of the supplementary work (1.7), under the condition $u = 0$, on S_u.

We formulate the solution by successive approximations in the form

$$\sigma_{ij}^{(r)} = \sigma_{ij0} + \sum_{s=1} c_{rs}\sigma_{ijs} \qquad (r = 0, 1, 2 \ldots), \tag{5.1}$$

where σ_{ij0} is a partial solution of the equilibrium equations, satisfying the given conditions on S_F; σ_{ijs} are partial solutions of the equilibrium equations, satisfying the zero boundary conditions of S_F and c_{rs} are arbitrary constants. Assuming $\bar{g}(T) = 1/G_0$, we find the zero approximation $\sigma_{ij}^{(0)}$, corresponding to the elastic problem, from the equation

$$\int\left(\frac{3}{2}k\sigma^2 + \frac{T^2}{2G_0}\right)dV = \min. \tag{5.2}$$

Coefficients c_{rs} obviously are determined from the system of linear nonuniform algebraic equations. Calculating the intensity T_0 from the stresses found $\sigma_{ij}^{(0)}$, we assume $G_1 = g(T_0/\Gamma_0)$, and we determine the first approximation $\sigma_{ij}^{(1)}$ from the conditions of minimization of the quadratic functional

$$\int\left(\frac{3}{2}k\sigma^2 + \frac{T^2}{2G_1}\right)dV = \min \tag{5.3}$$

etc. For the r-th approximation, we obtain

$$\int\left(\frac{3}{2}k\sigma^2 + \frac{T^2}{2G_r}\right)dV = \min. \tag{5.4}$$

We note that G_r also can be presented in the form (see Fig. 1)

$$G_r = \hat{G}_{r-1}\frac{T^*}{T_{r-1}}; \tag{5.5}$$

The presence of a variable G_r in the r-th approximation only complicates calculation of the quadrature somewhat; the r-th

approximation itself has the same form as for an elastic body.

It is advisable for concept (5.1) to contain the number of terms guaranteeing the necessary accuracy in solution of the elastic problem. The quadratures are conveniently found numerically. In determination of the "secant modulus," one may proceed directly from the experimental T-Γ curve. Preservation of the same form of solution in each approximation (only the coefficients c_{rs} are changed) considerably simplifies calculations and, in distinction from other methods of successive approximations, eliminates the bulk of the results.

We use a similar method in searching for the minimum energy in system (1.3). In this case, we find a solution of the problem u by successive approximations, in the form

$$u^{(r)} = u_0 + \sum_{s=1} c_{rs} u_s \quad (r = 0, 1, \ldots) \tag{5.6}$$

where u_0 satisfies the given conditions on S_u, u_s reverts to zero on S_u, and c_{rs} are arbitrary constants. In the zero approximation, we assume $g(\Gamma) = \text{const} = G_0$. In the r-th approximation, we assume $g(\Gamma) = g(\Gamma_{r-1})$

In the work of S. N. Roze [22], the convergence of the approximations formulated by the method set forth was studied. In this case, the same conditions were used, as were used earlier by A. Langenbach [13-14], in analysis of solvability of the Ritz system.

Solutions of the problems of elastic-plastic torsion of a rod of square cross section [32, 22], of the problem of elastic-plastic equilibrium of a nonuniformly heated, thin-walled pipe

under the action of internal pressure [33] and of the problem of elongation of a rectangular plate by nonuniformly distributed forces, were obtained by the modified Ritz method.

We note that finding the minimum at each stage can be accomplished by various methods. For example, the method of L. V. Kantorovich [23] of reduction to common differential equations can be used. Thus, in the plane case of problem (1.3), a solution can be found in the form

$$u^{(r)} = u_0 + \sum_{s=1} \Phi_s(x, y) f_{rs}(x)$$

where $\Phi_s(x,y)$ are known functions, reverting to zero at the boundary. For $f_{rs}(x)$, we obtain a system of common differential equations at each stage.

Another modification of the Ritz method was proposed by A. A. Il'yushin [24], applicable to variational equation (1.3). We designate the elastic potential by U, and we present relation (1.2) in the form

$$\sigma_{ij} = (1 - \omega) \frac{\partial U}{\partial \varepsilon_{ij}}, \qquad (5.7)$$

where $0 \leq \omega < 1$ is a known function. Then,

$$\Pi(\varepsilon) = U(\varepsilon) + \Pi_1(\varepsilon); \qquad \Pi_1(\varepsilon) = \int \omega$$

Searching for the minimum $\Pi^*(\varepsilon)$ in form (5.3), we arrive at the system of equations

$$\frac{\partial}{\partial c_{rs}} \left[\int U dV - \int \ldots dS \right] = - \int \ldots \qquad (5.8)$$

In the zero approximation (r = 0), we discard the right portions, and we determine the constants c_{0s} from the linear system of equations. Introducing the latter into the right parts of (5.8), we determine the first approximation of c_{js}, etc. It is clear that only the right parts of system (5.8) change each time.

A similar method can be developed in the problem of the minimum supplemental work $R^*(\sigma)$.

The conditions for convergence of the theories thus formulated to the Ritz approximation still have not been studied.

6. Optimum-Interpolation Method /183

The "deformation-stress" law usually contains some parameters, which essentially determine the mechanical properties of the medium. For example, for power law (1.2), this will be exponent m; in the linear hardening rule, the tangent of the slope of linear hardening, etc., is such a parameter. In expression (1.2), stress σ_{ij} does not depend on coefficient B, if u = 0 on part of surface S_u.

If a solution is determined comparatively easily and is known beforehand at certain values of the parameter, an approximate solution for other values of the parameter can be formulated, on the basis of the principle of minimum R^*.

Let us explain this by the example of the power law. At m = 1, we have a linear problem of the theory of elasticity; let the corresponding solution σ^1_{ij} be known.

As $m \to \infty$, we arrive at a limiting condition σ^∞_{ij}, which will coincide with the ideal-plastic distribution of stress in some cases and will not in others. The conditions for coincidence are indicated in work [4].

We seek a solution of variational equation (1.7) in the form

$$\sigma_{ij} = \sigma_{ij}^{\infty} + K_1(m)(\sigma_{ij}^{1} - \sigma_{ij}^{\infty}).$$

It is evident that σ_{ij} is a statically possible stress at any value of factor $K_1(m)$. From the principle of the minimum R*, we obtain the equation

$$\frac{\partial R^*}{\partial K_1} = 0,$$

determining $K_1(m)$; the latter changes from one to zero.

This method permits a solution of many problems to be formulated simply. Various example are presented in [4].

We note in conclusion that, if still other solutions are known, more precise approximations can be formulated. Let, say, still another solution for some $m = m_1$ be known. Then, a solution can be sought in the form

$$\sigma_{ij} = \sigma_{ij}^{\infty} + K_1(m)(\sigma_{ij}^{1} - \sigma_{ij}^{\infty}) + K_2(m)(\sigma_{ij}^{m_1} - \sigma_{ij}^{\infty}).$$

Factors K_1 and K_2 are determined from the equations

$$\frac{\partial R^*}{\partial K_1} = 0; \qquad \frac{\partial R^*}{\partial K_2} = 0.$$

7. Variational-Difference Method

Practical application of the network method for equations of the elliptical type involves well-known difficulties: formulation of the boundary conditions is very cumbersome, and it encompasses a considerable portion of the region, if the network is not too dense; the system of linear equations for the values of functions

in the nodes frequently is poorly determined and, sometimes (as a result of replacement of the differential operator by a finite-difference one), it has a determinant equal to zero [25].

R. Courant proposed another method of solution [26]. There are certain additions and theoretical comments on this method in the work of L. A. Oganesyan [25].

Let the variational problem contain derivatives of only the first order, with two independent variables x, y (for example, the torsion problem). We plot a triangular network on the x, y plane, and we will consider the values of the functions at the nodes of the network to be unknown. Replacing the integral surface by the surface of the corresponding polyhedron (Fig. 2), we calculate the values of the derivatives. In this case, the functional is reduced to a certain sum. The conditions of the minimum of this sum are reduced to a system of equations, which is close in form to the finite-difference system, but with different properties. This system of equations is better determined, since the positively determined operator remains unchanged, and only the search for the minimum is accomplished in a special class of functions. /184

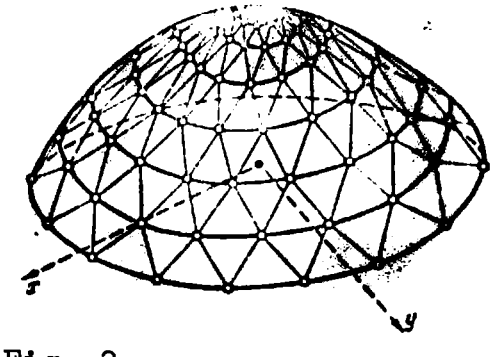

Fig. 2.

The system stated refers to quadratic functionals for second-order equations. Extension of this method to higher order equations involves certain difficulties and requires special methods of approximation of the integral surface in the supporting triangles, by breaking down into a system of second-order equations (for example, a fourth-order equation is broken down into a system of two second-order equations for two functions).

The method of R. Courant can be adapted for solution of variational problems of the theory of plasticity, if the solution is constructed in successive approximations. Let us explain this thought by the example of variational equation (1.7). Assuming $\bar{g}(T) = \text{const} = 1/G_0$, we find the zero approximation from the conditions of the minimum of quadratic functional (5.2). Calculating T_0 from the zero approximation and assuming $G_1 = g(T_0/\Gamma_0)$ we determine the first approximation from the conditions of the minimum of quadratic functional (5.3), etc.

Of course different variants in formulation of the approximation are possible, in particular, linearization by the system of A. A. Il'yushin (5.7).

Elastic-Plastic Hardening Body

1. Basic Relations

For simplicity, we assume that the loading surface is smooth and that hardening takes place. The rate of total deformation is composed of the rates of elastic deformation and plastic deformation $\xi_{ij} = \xi^e_{ij} + \xi^p_{ij}$.

The increments (or rates) of the deformation components are connected to the increments (or rates) of the stress components. According to Hooke's law $\xi^e_{ij} = c_{ijkl}\,\sigma_{kl}$, where c_{ijkl} are elastic constants. Further $\xi^p_{ij} = h_{ijkl}\,\sigma_{kl}$, where the coefficients h_{ijkl} are, generally speaking, complex functions of stress, deformation, deformation history, but not of rates σ_{ij}. In this manner

$$\xi_{ij} = (c_{ijkl} + h_{ijkl})\,\sigma_{kl}. \qquad (1.1)$$

These relations are outwardly analogous to functions for a linear-elastic anisotropic body; let them be solved relative to

the stress rates. The quadratic forms of the deformation rates $\tilde{\Pi}(\xi)$ and stress rates $\tilde{R}(\sigma)$ can be introduced, the coefficients of which depend, generally speaking, on deformation ε, stress σ and deformation history, in which

$$\dot{c}_{ij} = \frac{\partial \tilde{\Pi}(\xi)}{\partial \dot{\xi}_{ij}}, \qquad \dot{\xi}_{ij} = \frac{\partial \tilde{R}(\sigma)}{\partial \dot{\sigma}_{ij}}. \tag{1.2}$$

2. Variational Equations

Let velocity v be assigned to a portion of the surface of a body S_v and stress rate $\dot{F}_n$ on portion S_F.

By varying the field, a rate minimum takes place and /185

$$\delta\left[\int \tilde{\Pi}(\dot{\xi})\,dV - \int \dot{F}_n v\,dS_F\right] = 0. \tag{2.1}$$

By varying the equilibrium stress rates, we obtain

$$\delta\left[\int \tilde{R}(\dot{\sigma})\,dV - \int \dot{F}_n v\,dS_v\right] = 0. \tag{2.2}$$

In this manner, if small intervals of time are analyzed, for which coefficients of the form $\tilde{\Pi}$ and $\tilde{R}$ can be considered constant, in these intervals, for velocity v, we have a problem of the minimum of quadratic functional (2.1) and, for the stress rates $\dot{\sigma}_{ij}$, the problem of the minimum of quadratic functional (2.2).

In each of the intervals of time, the boundary value problem can be solved by the Ritz method in the normal form (on the basis of the first or second variational principle). Correcting the

values of the coefficients in each succeeding interval and moving by steps, a complete solution can be constructed.

This method is close to the method used successfully by V. I. Feodos'yev [27], in analysis of the problem of stability beyond the elastic limit. The solution is found in the following form: $v_x = A_i v_{xi}$; $v_y = B_i v_{yi}$; $v_z = C_i b_{zi}$, where v_{xi}, v_{yi} and v_{zi} are coordinate functions and A_i, B_i and C_i are parameters dependent on time. Calculating the stress rates in conformance with (1.2), we introduce them into the differential equations of motion. Further, applying the method of Galerkin to the resulting equations (under conditions of orthogonality of the coordinate functions), we obtain a system of common differential equations relative to parameters A_i, B_i and C_i, with known initial data. The Cauchey problem is solved by computer for this system.

Of course, a solution can be constructed from the loading parameter (without taking account of the inertial terms).

Rigid-Plastic Body: Application to Theory of Pressure Treatment of Metals

1. Extreme Principles

In the case of an ideal rigid-plastic body (we confine ourselves to examination of the medium of St. Venan-Mizes, with the flow condition of Mizes), the components of the stress deviator are single-value functions of the components of the deformation rate (the reverse is not true). Then,

$$L(\xi) = \tau_s H, \tag{1.1}$$

where $H = (2\xi_{ij}\xi_{ij})^{1/2}$ is the intensity of the displacement deformation rate and τ_s is the yield point during displacement. Additional scattering Λ was not determined.

The first extreme principle follows (in transition to the velocities) from (1.3, p. 3)

$$L^*(\xi) \equiv \tau_s \int H dV - \int F_n v dS_F = \min, \tag{1.2}$$

in which the extreme here also is analytical. In the case of a rigid-plastic body, the integration can be considered to be distributed over the entire volume of the body. Possible discontinuities in the fields of stress (on certain surfaces S_i) and rates (on surfaces S_j) must be taken into account. The first discontinuities, as is well-known, do not affect formulation of the extreme principles. In the case of a discontinuity field, the rates in expression $L^*(\xi)$ should include scattering, producible on the surfaces of the discontinuity,

$$\tau_s \sum_j \int |v_t| \cdot \tag{1.3}$$

where $|v_t|$ is the jump in the tangential component of the velocity vector on S_j.

In the case being examined, as has already been pointed out, supplemental scattering Λ was not determined. As follows from a graphical representation of scattering L and supplemental scattering Λ [4], the latter can be considered to be equal to zero. Changing in (1.7, p. 3) to velocities v and assuming $\Lambda = 0$, we formally arrive at the principle of the maximum scattering for actual surface forces

$$\int F_n v dS_v = \max$$

or, in more general form,

$$\int F_n v dS_v \geq \int F'_n v dS_v \qquad (1.4)$$

where forces F'_n correspond to any statically possible stress, lying within the circle of the flow or on it. If the field is discontinuous, the corresponding term must be added to each side of inequality (1.4).

Each kinematically possible velocity field v' leads to an upper boundary for the limiting load. On the other hand, each statically possible stress lying within the flow circle or on it gives a lower boundary for the limiting load.

These results are the basis of various methods for approximate determination of the limiting loads. For example, we assign a velocity field, so that it depends on a series of indefinite parameters; then, the upper boundary of the limiting load is a function of these parameters; selecting parameter values from the condition of the minimum limiting load, we obtain for the latter a more or less acceptable estimate from above. Applying such a method with a given stress field, we find the optimum lower estimate. Synthesis of a suitable kinematically possible velocity field over the entire body, as a rule, is considerably simpler than synthesizing the permissible stress field. Selection of the latter is hampered by the same limitations, to which this field must be subject (conditions of equilibrium in V and on S_F; conditions of plasticity) over the entire volume of the body. Besides, it is not always easy to foresee the picture of the stress. In connection with this, it turns out to be more difficult to obtain a good lower estimate of the limiting load than it does the upper one.

2. Application to Theory of Pressure Treatment of Metals

In recent years, there has been a considerable development in variational methods of solution of problems of the theory of pressure treatment of metals. This has been facilitated mainly by two circumstances. First, the configuration of the body can be extremely complicated, in connection with which, the introduction of other methods is difficult. Second, the requirements for accuracy are low, which permits confining oneself to one or two arbitrary parameters in using variational methods. Let us briefly examine here some works, without sticking to chronological order.

In the works of G. Ya. Gun and P. I. Polukhin [28-29], a given steady plastic flow was examined, under conditions of plane deformation. As a kinematically possible velocity field, it is proposed to adopt the velocity field in the corresponding problem of vortex-free flow of an ideal, incompressible fluid. If $w(z)$ is a complex potential, $z = x + iy$, the complex velocity equals

$$v$$

and the intensity of the displacement deformation rate

$$2|w''(z)|.$$

Scattering of the body is determined by the power of external forces; the latter are composed of the power of the force of friction (assumed equal to $\mu\tau_s$, $\mu \leq 1$) on the "adhesion" sections

$$-\tau_s \int \mu v_t dc,$$

where v_t is the tangential component of the velocity on contour c and the power of the moving instrument (considered to be smooth) pv_0, where p is the average normal pressure, and v_0 is the flow velocity. In this manner,

$$p = \frac{\tau_s}{v_0}\left[2\int\int \omega \, dx\,dy - \int v_t\, dc\right].$$

Integration is carried out over the area of the "focus of deformation." If the complex potential contains some indeterminate parameters c_k, finding them from the conditions of the minimum p, we obtain the closest value of the force. For synthesis of the flow sought, the well developed methods of the plane problem of hydrodynamics, in particular, the method of synthesizing jet flows, are used.

A large number of diverse problems of the theory of pressure treatment of metals have been studied in numerous works of I. Ya. Tarnovskiy and his colleagues. These works are summed up in a recently published monograph [30], in which the approximate solutions of the problem of free forging, stamping, rolling and wire drawing are stated in detail. To become familiar with the procedures which are used in this case, let us examine the problem of settling of a round cylindrical billet between polishing plates (Fig. 3).

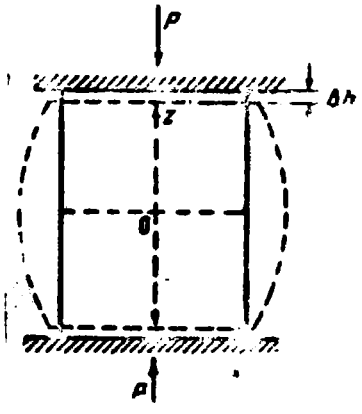

Fig. 3.

On the assumption of smallness of the deformation, the problem is solved on the basis of the principle of the minimum total energy. Displacement along the z axis is assigned in the form

$$u_z = -\left[az + c_1 z\left(1 - \frac{z^2}{h^2}\right) + c_2 z\,\frac{r^2}{R^2}\left(1 - \frac{z^2}{h^2}\right)\right],$$

where c_1 and c_2 are arbitrary constants and $\varepsilon = -\Delta h/h$ is considered to be fixed. The relative elongation ε_2 is determined by differentiation.

From the condition of incompressibility,

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \varepsilon_z = 0$$

we find u_r and we further calculate the deformation components ε_r, ε_ϕ, and γ_{rz}.

Constants $c_i (i = 1, 2)$ are determined from the conditions of the minimum total energy

$$\frac{\partial \Pi}{\partial c_i} \equiv \frac{\partial}{\partial c_i}\left[\tau_s \int_0^{2\pi}\int_0^h\int_0^R \Gamma r\, dr\, d\varphi\, dz + \mu \int_0^{2\pi}\int_0^R (u_r)_{z=h} r\, dr\, d\varphi\right] = 0.$$

In the latter, a quite rough approximation procedure is used (as in many other problems), connected with the inequality of Bunyakovskiy-Schwartz. Namely, it is considered that /188

$$\int \Gamma dV \approx \sqrt{V \int \Gamma^2 dV}.$$

The preceding equations then take the form

$$\frac{1}{2\Gamma_c} \int_0^{2\pi}\int_0^h\int_0^R \frac{\partial \Gamma^2}{\partial c_i} r\, dr\, d\varphi\, dz + \mu \int_0^{2\pi}\int_0^R \frac{\partial}{\partial c_i}(u_r)_{z=h} r\, dr\, d\varphi = 0, \qquad (2.1)$$

where it is assumed that

$$\Gamma_c = \left[\frac{1}{V}\int_0^{2\pi}\int_0^h\int_0^R \Gamma^2 r\, dr\, d\varphi\, dz\right]^{1/2}.$$

The value of Γ_c is determined autonomously from the condition that the cylinder settles uniformly; then, $\Gamma_c = \sqrt{3}\,\bar{\varepsilon}$. Here, equations (2.1) are a system of two linear, nonuniform equations relative to c_1/ε and c_2/ε. Finally, the settling force P is found from the equality of work $P\Delta h$ to the work of all "resistances to deformation" $\bar{\Pi}$.

By means of a rough method of averaging certain values of displacement u_r and subsequent integration, the authors arrive at a formula for the final configuration of the billets with great settling. The solution is compared with experimental data.

From the works set forth above, the mechanical properties of the metals are taken into account insufficiently fully. Hardening actually is estimated by correction of the value of the yield point. In the recently published work of R. Hill [31], somewhat more general variational equations are formed.

It is assume that the flow surface is smooth, isotropic and independent of the average pressure; its shape is determined by the temperature and deformation history. The flow along the normals to the surface are not assumed, but the stress is connected in a single way with the direction of the deformation increment.

The body being examined contains a deformable volume ("focus of deformation") V, with surface S and undeformable zones. Surface S consists of three parts $S = S_\phi + S_c + S_r$, where S_ϕ is determined by the shape of the instrument (matrix), S_c is the free surface and S_r is the interface with the rigid zones. Kinematic conditions are assigned to S_r and, perhaps, conditions of static equivalence. The surface force $P_n = 0$ on S_c. A tangential stress component τ_n is assigned to S_ϕ; thus, for a smooth matrix $\tau_n = 0$, for a rough one $\tau_n = k$, where k is the local yield point, etc.

The actual stress σ_{ij} is satisfied by the differential equations of equilibrium in volume V and by the given conditions on S.

We select a kinematically possible approximating velocity field v_i; we calculate the stress field σ'_{ij} from this field, on the basis of the flow relationship; obviously the latter will not be equilibrium.

We take a kinematically possible derivative velocity field w_i (making the velocity field orthogonal). The fields selected should have equal arbitrariness (equal to the "dimensionality"); in particular, they can coincide.

From the principle of possible velocities, we have

$$\int \sigma_{ij} \frac{\partial w_i}{\partial x_j} dV = \int F_{ni} w_i dS$$

where F_{ni} are the components of the surface force. For simplicity of presentation, we examine a case of continuous fields.

When fields v_i and w_i are selected, the usual methods of variational calculus for derivation of the "equation of Euler" and the corresponding natural boundary conditions are applied. /189

As an illustration, let us examine the example of settling of a cylinder of arbitrary cross section between polishing plates (see Fig. 3); coulumb friction $\mu\sigma_n$ acts on the contact planes.

We assign the following approximating fields:

$$\begin{aligned} v_x &= \frac{1}{2} x + \mu \psi(x, y) \varphi'(z); \\ v_y &= \frac{1}{2} y + \mu \theta(x, y) \varphi'(z); \\ v_z &= -z + \mu \theta(x, y) \varphi(z), \end{aligned} \qquad (2.2)$$

where $\phi(h) = 0$; it follows from the equation of incompressibility that $\theta = -\left(\frac{\partial \alpha}{\partial x} + \right)$. We assume the tangential stress τ on the bases is approximately constant and directed along the radius; then $\alpha = x/r$; $\beta = y/r$.

Functions $\phi(z)$ are found. We take the orthogonalizing field as:

$$\begin{aligned} w_x &= \tfrac{1}{2} x\psi'(z); \\ w_y &= \tfrac{1}{2} y\psi'(z); \\ w_z &= -\psi(z), \end{aligned} \qquad (2.3)$$

where $\psi(z)$ is an arbitrary function. Introducing (2.3) into equation (2.1) and applying normal transformations, by virtue of the arbitrariness of $\psi'(z)$, we obtain the equilibrium equation

$$\int r\tau_{rz} dS_0 + 2(P - \sigma_s S_0) z = 0, \qquad (2.4)$$

where S_0 is the cross section area and the conditions on the bases

$$\int r(\tau - \mu\sigma_s) dS_0 = 0. \qquad (2.5)$$

Normal pressure σ_n is assumed to be equal to the yield point σ_s, in the first approximation.

Applying the law of plastic flow of Mizes, we calculate the stress components from the approximating field (2.2) and we obtain from (2.4) the differential equation

$$\frac{1}{\beta}\varphi'' + \frac{1}{a^2}\varphi + \frac{b}{h}z = 0,$$

where a, b are certain coefficients.

In this manner, in the work of Hill, a variational method is proposed for derivation of "engineering equations" and the corresponding boundary conditions of the problem, resting on the choice of a suitable approximating velocity field.

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